

# On two exactly-solvable one-dimensional Hamiltonians with PT symmetry

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We provide an explanation to the behaviour of the spectra of two exactly-solvable one-dimensional Hamiltonians with PT symmetry proposed earlier. We calculate the branch points at which pairs of eigenvalues coalesce and discuss the perturbation series.

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## I. INTRODUCTION

Some time ago Ahmed[1] solved the Schrödinger equation with the PT-symmetric potentials  $V_1(x) = x^2/4 + ig|x|x$  and  $V_2(x) = |x| + igx$  exactly and found an interesting behaviour of the corresponding point spectra. In both cases there is an infinite number of real eigenvalues when  $g = 0$  and as  $g$  increases the number of real eigenvalues decreases to just one. At every finite nonzero value of  $g$  there is an odd number of real eigenvalues (say  $2k+1$ ) and as  $g$  increases this number reduces to  $2k-1, 2k-3, 2k-5, \dots, 1$ . Ahmed did not explain this phenomenon satisfactorily and did not identify the critical  $g$  values at which each change takes place. However, he proposed to call it *scarcity of real discrete eigenvalues*.

The purpose of this paper is to provide a more detailed discussion of the behaviour of the spectra of those models. In section II we outline some properties of the perturbation series for a particular class of parameter-dependent Hamiltonians and in section III we discuss the spectra of the models mentioned above. Finally, in section IV we draw conclusions.

## II. PERTURBATION SERIES

Let  $H(\lambda)$  be a parameter-dependent linear operator and  $U$  a linear invertible operator such that  $UH(\lambda)U^{-1} = H(-\lambda)$ . Therefore, if  $\psi_n(\lambda)$  is an eigenfunction of  $H(\lambda)$  with eigenvalue  $E_n(\lambda)$  then it follows from

$$UH(\lambda)\psi_n(\lambda) = UH(\lambda)U^{-1}U\psi_n(\lambda) = H(-\lambda)U\psi_n(\lambda) = E_n(\lambda)U\psi_n(\lambda), \quad (1)$$

that  $U\psi_n(\lambda)$  is proportional to an eigenfunction  $\psi_m(-\lambda)$  of  $H(-\lambda)$  with eigenvalue

$$E_m(-\lambda) = E_n(\lambda). \quad (2)$$

Here, we are interested in the case that  $H(0)$  is an Hermitian operator. Since equation (2) is assumed to be valid for all  $\lambda$  we conclude that  $E_m(0) = E_n(0)$ . Consequently, if the eigenvalues of  $H(0)$  are nondegenerate then  $m = n$  and  $E_n(\lambda) = E_n(-\lambda)$  for all  $\lambda$ .

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If the eigenvalue  $E_n(\lambda)$  can be expanded in a Taylor series about  $\lambda = 0$

$$E_n(\lambda) = \sum_{j=0} E_n^{(j)} \lambda^j, \quad (3)$$

then  $E_n^{(2j+1)} = 0$  for all  $j = 0, 1, \dots$ . If this expansion converges for  $|\lambda| < R_n$  then

$$E_n(ig) = \sum_{j=0} E_n^{(2j)} (-g^2)^j, \quad (4)$$

is real for all  $|g| < R_n$  provided that  $g$  and the perturbation coefficients  $E_n^{(2j)}$  are real.

### III. TWO EXACTLY-SOLVABLE PT-SYMMETRIC MODELS

As indicated in the Introduction Ahmed[1] briefly discussed two exactly-solvable one-dimensional PT-symmetric models. One of them is given by

$$H(\lambda) = p^2 + \frac{1}{4}x^2 + \lambda|x|x, \quad (5)$$

where,  $[x, p] = i$ . In this case the parity transformation

$$PpP = -p, \quad PxP = -x, \quad (6)$$

yields  $PH(\lambda)P = H(-\lambda)$ , where  $P^{-1} = P$ . This Hamiltonian operator satisfies the conditions discussed in the preceding section and is PT-symmetric when  $\lambda = ig$ ,  $g$  real, because  $V(-x)^* = V(x)$  (see, for example, [2] and references therein).

Ahmed[1] proved that when  $\lambda = ig$  the eigenvalues are solutions to the nonlinear equation

$$D(E, g) = \Re \left[ \frac{(\frac{1}{4} - ig)^{1/4}}{\Gamma\left(\frac{3}{4} - \frac{E}{2\sqrt{1+4ig}}\right) \Gamma\left(\frac{1}{4} - \frac{E}{2\sqrt{1-4ig}}\right)} \right] = 0, \quad (7)$$

and from a straightforward numerical calculation he conjectured that the number of real eigenvalues decreases from infinity ( $g = 0$ ) to just one (for a sufficiently large value of  $g$ ).

The reason for this behaviour is that as  $g$  increases from  $g = 0$  a pair of eigenvalues  $E_{2n-1}$  and  $E_{2n}$ ,  $n = 1, 2, \dots$ , approach each other, coalesce at  $g = g_n > 0$  and become a pair of complex conjugate numbers for  $g > g_n$ . The only eigenvalue that appears to remain real for all  $g$  is  $E_0$ . This result is consistent with the fact that there is just one real eigenvalue when  $V(x) = i|x|x$ [3] that is the potential for the strong-coupling limit ( $g \rightarrow \infty$ ) of (5). In fact, from the canonical transformation

$$U_\gamma x U_\gamma^{-1} = \gamma x, \quad U_\gamma p U_\gamma^{-1} = \gamma^{-1} p \quad (8)$$

with  $\gamma = g^{-1/4}$  we easily prove that

$$\lim_{g \rightarrow \infty} g^{-1/2} U_\gamma x U_\gamma^{-1} = p^2 + i|x|x. \quad (9)$$

Numerical calculation shows that the critical points  $g_n$  decrease with  $n$  so that for a given value of  $g$  only the pairs of eigenvalues with  $g_n < g$  are real. When  $g > g_1$  there is only one real eigenvalue ( $E_0$ ) and when  $g_{n+1} < g < g_n$

there are  $2n + 1$  real eigenvalues. This fact explains why Ahmed[1] obtained an odd number of real eigenvalues for every  $g < g_1$ . For example, Fig. 1 illustrates the coalescence of the eigenvalues  $E_1$  and  $E_2$ . Ahmed[1] suggests that there is just one real eigenvalue for  $g = 2$ ; however, our calculation shows that there are three real eigenvalues:  $E_0 = 1.720857958$ ,  $E_1 = 6.579362071$  and  $E_2 = 7.39812626$ . He probably missed the two eigenvalues  $E_1$  and  $E_2$  because they are quite close to each other or due to insufficient accuracy in the calculation.

Equation (7) yields either  $E(g)$  or  $g(E)$ . If we take into account that  $dD/dE = 0$  when  $E$  and  $g$  are linked by  $D(E, g) = 0$  then we find that

$$\frac{dg}{dE} = \frac{\partial D / \partial E}{\partial D / \partial g}. \quad (10)$$

Therefore, the coalescence points that are given by  $dg/dE = 0$  (square-root branch points) can be easily obtained by solving the system of nonlinear equations (see, for example, [4, 5])

$$D(E, g) = 0, \quad \frac{\partial D}{\partial E}(E, g) = 0. \quad (11)$$

Table I shows the first five critical points (note that  $g_1 > 2$ ).

As argued in the preceding section, the application of perturbation theory to the model (5) should yield a  $g^2$ -series for every eigenvalue  $E_n(ig)$ . Since both  $H(0)$  and  $dH(\lambda)/d\lambda = |x|x$  are real, then the coefficients  $E_n^{(j)}$  are real. The first three terms of the perturbation series for the first three eigenvalues are:

$$\begin{aligned} E_0(ig) &= \frac{1}{2} + (2 + \ln 2) g^2 + \\ &\quad \left[ -7 \ln 2 + 3 (\ln 2)^2 + \frac{1}{3} (\ln 2)^3 - \frac{1}{4} \zeta(3) - 12 + \frac{1}{12} \pi^2 \ln 2 - \frac{1}{12} \pi^2 \right] g^4 + \dots \\ &= 0.5000000000 + 2.693147181 g^2 - 15.85255355 g^4 + \dots \end{aligned} \quad (12)$$

$$\begin{aligned} E_1(ig) &= \frac{3}{2} + (3 + 9 \ln 2) g^2 + \\ &\quad \left[ -\frac{81}{4} \zeta(3) - 63 \ln 2 + 54 (\ln 2)^2 - 18 + \frac{27}{4} \pi^2 \ln 2 + 27 (\ln 2)^3 - \frac{9}{2} \pi^2 \right] g^4 + \dots \\ &= 1.5000000000 + 9.238324625 g^2 - 49.30966898 g^4 + \dots \end{aligned} \quad (13)$$

$$\begin{aligned} E_2(ig) &= \frac{5}{2} + \left( -\frac{5}{2} + 25 \ln 2 \right) g^2 + \\ &\quad \left[ -\frac{625}{4} \zeta(3) + 75 \ln 2 + \frac{125}{2} (\ln 2)^2 + \frac{625}{12} \pi^2 \ln 2 - \frac{405}{8} + \frac{625}{3} (\ln 2)^3 - \frac{875}{24} \pi^2 \right] g^4 + \dots \\ &= 2.5000000000 + 14.82867952 g^2 - 90.5745397 g^4 + \dots \end{aligned} \quad (14)$$

where  $\zeta(k)$  is the zeta function. The radius of convergence of the perturbation series for the pair of eigenvalues  $E_{2n-1}$  and  $E_{2n}$ ,  $n > 0$  is expected to be  $g_n$ .

The eigenvalues for the second model

$$H(\lambda) = p^2 + |x| + \lambda x, \quad (15)$$

when  $\lambda = ig$ , are roots of the nonlinear expression[1]

$$\begin{aligned} D(E, g) &= \nu_1 \mu_2 Ai(E/\mu_2) Ai'(E/\mu_1) - \mu_1 \nu_2 Ai(E/\mu_1) Ai'(E/\mu_2) = 0, \\ \nu_1 &= -1 + ig, \nu_2 = 1 + ig, \mu_1 = -(\nu_1^2)^{1/3}, \mu_2 = -(\nu_2^2)^{1/3}. \end{aligned} \quad (16)$$

The behaviour of the spectrum of the PT-symmetric Hamiltonian (15) is similar to the one discussed above in all relevant aspects except one. In this case the strong-coupling limit is given by the canonical transformation (8) with  $\gamma = g^{-1/3}$  and

$$\lim_{g \rightarrow \infty} g^{-2/3} U_\gamma H U_\gamma^{-1} = p^2 + ix. \quad (17)$$

Bender and Boettcher[6] proved that this model does not exhibit eigenvalues and that the ground state of  $H = p^2 - (ix)^N$  diverges as  $N \rightarrow 1^+$ . This is the most relevant difference with respect to the preceding example.

The first four critical points of the spectrum of (15) are shown in Table II.

#### IV. CONCLUSIONS

The two PT-symmetric Hamiltonians discussed in the preceding section exhibit critical points  $g_n > 0$  that decrease with  $n$  (we omit the consideration of the case  $g < 0$  because  $E(-ig) = E(ig)$ ). Since it appears that  $\lim_{n \rightarrow \infty} g_n = 0$  the number of real eigenvalues is finite for all  $g > 0$ . This result is important because it may shed light on the spectra of a family of PT-symmetric multidimensional oscillators with critical points that decrease with the magnitude of the coalescing eigenvalues[7, 8]. Present results suggest that the PT-phase transition[7] for the Hamiltonians (5) and (15) takes place at the trivial Hermitian limit  $g = 0$ . Such a behaviour commonly takes place in multidimensional problems that exhibit some kind of point-group symmetry[8–11]. For this reason the results above for one-dimensional models that merely exhibit parity symmetry  $PH(0)P = H(0)$  appear to be quite interesting.

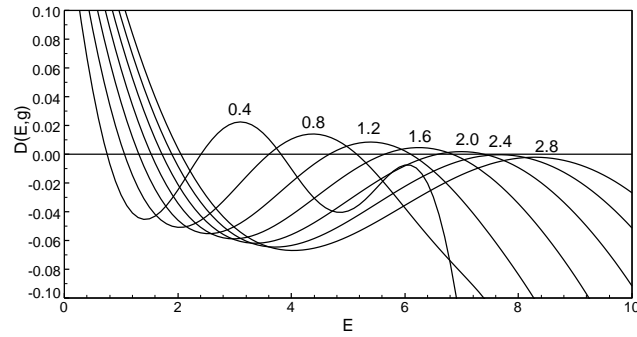
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TABLE I: First critical points for the Hamiltonian (5)

$n$	$g_n$	$E_{2n-1} = E_{2n}$
1	2.3262718829432516237	7.5466636499753202587
2	0.38031593803698939339	5.9543772352940988301
3	0.24202011969988939326	7.5185674146488458902
4	0.18511586549014097563	9.3051810150783521871
5	0.15263064129841349291	11.167144554229643582

TABLE II: First critical points for the Hamiltonian (15)

$n$	$g_n$	$E_{2n-1} = E_{2n}$
1	0.62782075783846150477	3.3482781381164933022
2	0.35319223654461721210	4.7569117414811876143
3	0.25419305000674157648	6.0526901706522118370
4	0.20120031566314106331	7.2414114087019937638

FIG. 1:  $D(E, g)$  vs.  $E$  for  $g = 0.4, 0.8, 1.2, 1.6, 2.0, 2.4, 2.8$